Commensurable indicators - finding potentially metrically invariant indicators

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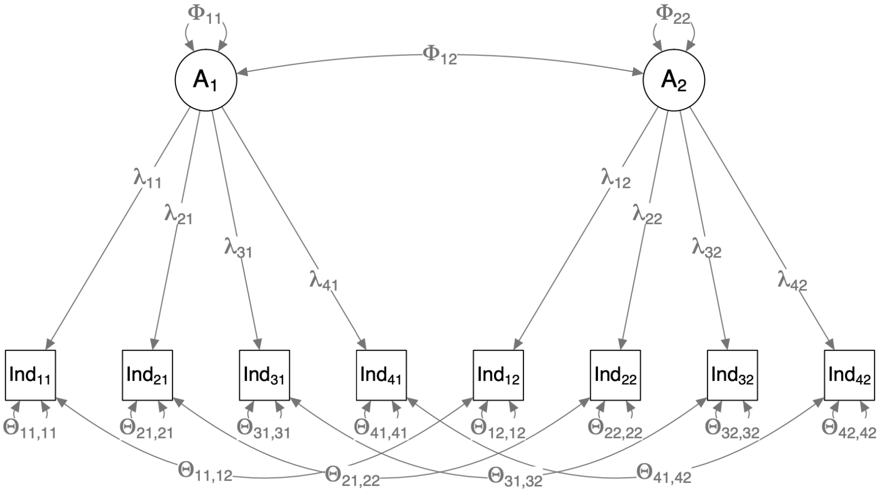
Example longitudinal CFA



Example longitudinal CFA

Loadings	Time 1	Time 2
λ_{1l}	0.80	5.00
λ_{2l}	4.00	5.00
λ_{3l}	3.20	4.00
λ_{4l}	2.00	2.00
Lat. var.	1.00	1.00
Lat. cov	0.24	
$\theta_{1l,1l}$	3	2
$\theta_{2l,2l}$	1	7
$\theta_{3l,3l}$	4	1
$\theta_{4l,4l}$	2	8
$\theta_{11,12}$	0.20	
$\theta_{21,22}$	0.50	
$\theta_{31,32}$	0.25	
$\theta_{41,42}$	0.50	

- Sample with $N = 500$ cases
- Does the configural model hold?
- Result:
  $\chi^2(15) = 7.045$, $p = 0.956$,
  CFI= 1.000, RMSEA= 0.000
- Does full metric MI hold?
- Result:
  $\chi^2(18) = 637.553$, $p < .001$,
  CFI= 0.800, RMSEA= 0.262
- Full metric MI does not hold, but are there invariant indicators so that a partial metric MI holds?

DGP - Notation

Data generating process (DGP) and estimated model:

- We distinguish between a DGP, a specified model, and an estimated model (Klopp & Klößner, 2023, p. 197-200).

Notation:

- The number of indicators is p, and index $j \in \{1, \ldots, p\}$ denotes the indicators.
- The number of factors is m, and index $i \in \{1, \ldots, m\}$ denotes the factors.
- The number of time points is L, and index $l \in \{1, \ldots, L\}$ denotes the time points.
- The loadings (and the other parameters) follow the usual conventions.
- The parameter vector is called $\mathbf{p}$, or simply parameter $\mathbf{p}$.

Definition - Set of indicators

Definition (Set of indicators)

For a given parameter $\mathbf{p}$ and for all factors i,

$$\text{Ind}_i(\mathbf{p}) = \{j : \exists l \text{ with } \lambda_{l,ji} \neq 0\}$$

is called the *set of indicators*, i.e., the indicators which load on factor i for at least one instance.

Loadings	Time 1	Time 2
λ_{1l}	0.80	5.00
λ_{2l}	4.00	5.00
λ_{3l}	3.20	4.00
λ_{4l}	2.00	2.00

- Set of indicators: $\text{Ind}_1(\mathbf{p}) = \{1, 2, 3, 4\}$

Definition - Loading profile

Definition (Loading profile)

For a given parameter $\mathbf{p}$, all factors i and all indicators $j \in \text{Ind}_i(\mathbf{p})$, the *loading profile* of an indicator j on factor i is the non-zero vector

$$\lambda_{ji}^{*}(\mathbf{p}) := (\lambda_{1,ji}, \ldots, \lambda_{L,ji})$$

Loadings	Time 1	Time 2
λ_{1l}	0.80	5.00
λ_{2l}	4.00	5.00
λ_{3l}	3.20	4.00
λ_{4l}	2.00	2.00

- Loading profiles:
  $\lambda_{11}(\mathbf{p}) = (0.80, 5.00)$
  $\lambda_{21}(\mathbf{p}) = (4.00, 5.00)$
  $\lambda_{31}(\mathbf{p}) = (3.20, 4.00)$
  $\lambda_{41}(\mathbf{p}) = (2.00, 2.00)$

Definition - Metric Invariance

Definition (Metric invariance)

For a given parameter $\mathbf{p}$, *metric invariance* of indicator j with respect to factor i and parameter $\mathbf{p}$ is given if the loading profile $\lambda^*_{ji}(\mathbf{p})$ is a multiple of the vector of ones $\mathbf{1}$.

Loadings	Time 1	Time 2
λ_{1l}	0.80	5.00
λ_{2l}	4.00	5.00
λ_{3l}	3.20	4.00
λ_{4l}	2.00	2.00

- Metric invariance: $\lambda_{41}(\mathbf{p}) = (2.00, 2.00) = 2 \cdot \mathbf{1}$

Definition - Commensurability - Lemma

Definition (Commensurability)

For a given parameter $\mathbf{p}$ and for all factors i and all indicators $j_1, j_2 \in \text{Ind}_i(\mathbf{p})$, $\sim$ indicates the *relation of commensurability* between indicators j_1 and j_2 with respect to factor i and parameter $\mathbf{p}$, with $j_1 \sim j_2$ if and only if $\lambda_{j_1 i}(\mathbf{p})$ and $\lambda_{j_2 i}(\mathbf{p})$ are non-zero multiples of each other.

Loadings	Time 1	Time 2
λ_{1I}	0.80	5.00
λ_{2I}	4.00	5.00
λ_{3I}	3.20	4.00
λ_{4I}	2.00	2.00

- Commensurable indicators: $\lambda_{31}(\mathbf{p}) = (3.20, 4.00) = 0.8 \cdot (4, 5) = 0.8 \cdot \lambda_{21}(\mathbf{p})$

Lemma (Equivalence relation)

The relation of commensurability is an equivalence relation, i.e., it is reflexive, symmetric, and transitive.

Definition - CIS - PCIS

Definition (CIS and PCIS)

The equivalence class of commensurability relations is called *commensurable indicator subset* CIS with respect to factor i and parameter $\mathbf{p}$. The partition of $\mathrm{Ind}_i(\mathbf{p})$ given by the CIS is called the partition of commensurable indicator subsets $\mathrm{PCIS}_i(\mathbf{p})$.

$$
\begin{aligned}
C_1 &= \{1\} \\
C_2 &= \{2, 3\} \\
C_3 &= \{4\} \\
\mathrm{PCIS}_1(\mathbf{p}) &= \{\{1\}, \{2, 3\}, \{4\}\}
\end{aligned}
$$

Definition - Average loading profile - Static indicator loading

Definition (Average loading profile and Static indicator loading)

For a commensurable indicator subset $C \in \mathrm{PCIS}_i(\mathbf{p})$, the *average loading profile* $\overline{\lambda^*_{C,i}(\mathbf{p})}$ of C is defined as

$$\overline{\lambda^*_{C,i}(\mathbf{p})} := \frac{1}{|C|} \sum_{j \in C} \lambda^*_{ji}(\mathbf{p}),$$

and the *static indicator loading* $\lambda^*_{j,C}(\mathbf{p})$ of C with respect to indicator $j \in C$ is defined as

$$\lambda^*_{j,C}(\mathbf{p}) := \frac{\mathbf{1}^\top \lambda^*_{ji}(\mathbf{p})}{\mathbf{1}^\top \overline{\lambda^*_{C,i}(\mathbf{p})}}.$$

Example

	CIS	$\lambda^*_{ji}(\mathbf{p})$		$\mathbf{1}^\top \cdot \lambda^*_{ji}(\mathbf{p})$	$\overline{\lambda^*_{C,i}(\mathbf{p})}$		$\mathbf{1}^\top \cdot \overline{\lambda^*_{C,i}(\mathbf{p})}$	$\lambda^*_{j,c}(\mathbf{p})$
Ind. 1	C_1	0.80	5.00	5.80	0.80	5.00	5.80	1
Ind. 2	C_2	4.00	5.00	9.00	3.60	4.50	8.10	10/9
Ind. 3	C_2	3.20	4.00	7.20				8/9
Ind. 4	C_3	2.00	2.00	4.00	2.00	2.00	4.00	1

- Average loading profile of a CIS C:

$$\overline{\lambda^*_{C,i}(\mathbf{p})} := \frac{1}{|C|} \sum_{j \in C} \lambda^*_{ji}(\mathbf{p})$$

- Static indicator loading of CIS C with respect to indicator $j \in C$:

$$\lambda^*_{j,C}(\mathbf{p}) := \frac{\mathbf{1}^\top \lambda^*_{ji}(\mathbf{p})}{\mathbf{1}^\top \overline{\lambda^*_{C,i}(\mathbf{p})}}$$

Proposition

For all factors i, parameters $\mathbf{p}$, a CIS $C \in \text{PCIS}_i(\mathbf{p})$:

1. The loading profile can be written as the product of the indicator's static loading on C and the average loading profile of C
$$\lambda^*_{ji}(\mathbf{p}) = \lambda^*_{j,C}(\mathbf{p}) \cdot \overline{\lambda^*_{C,i}(\mathbf{p})}.$$
2. Within a CIS C, the static indicator loadings on that CIS average to unity
$$\frac{1}{|C|} \sum_{j \in C} \lambda^*_{j,C}(\mathbf{p}) = 1.$$
3. Metric invariance with respect to indicator j is given if and only if the average loadings profile of a CIS C is a non-zero multiple of $\mathbf{1}$, the vector of ones. In this case, we say that a CIS C fulfills metric invariance.

Proposition - Example

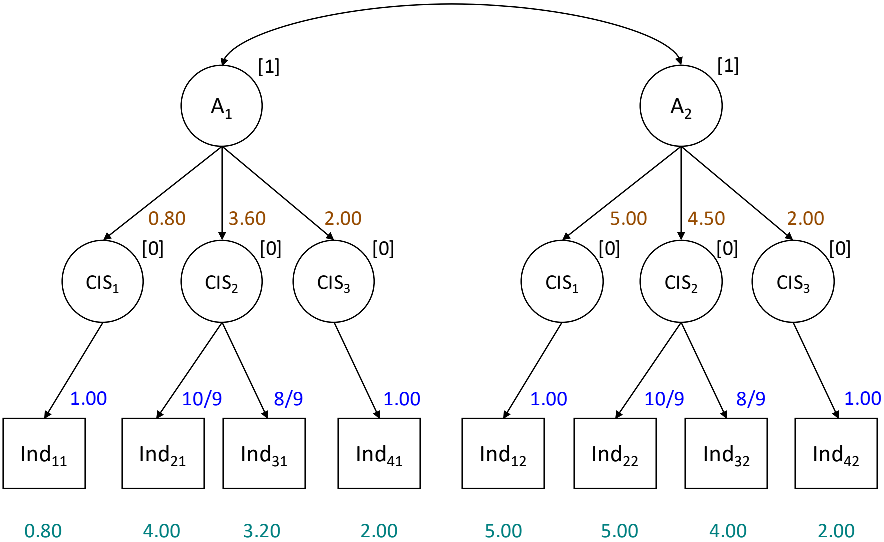
Decomposition

$$\lambda^*_{ji}(\mathbf{p}) = \overline{\lambda^*_{C,i}(\mathbf{p})} \cdot \lambda^*_{j,C}(\mathbf{p}).$$

	CIS	$\lambda^*_{ji}(\mathbf{p})$		$\mathbf{1}^\top \cdot \lambda^*_{ji}(\mathbf{p})$	$\overline{\lambda^*_{C,i}(\mathbf{p})}$		$\mathbf{1}^\top \cdot \overline{\lambda^*_{C,i}(\mathbf{p})}$	$\lambda^*_{j,c}(\mathbf{p})$
Ind. 1	C_1	0.80	5.00	5.80	0.80	5.00	5.80	1
Ind. 2	C_2	4.00	5.00	9.00	3.60	4.50	8.10	10/9
Ind. 3	C_2	3.20	4.00	7.20				8/9
Ind. 4	C_3	2.00	2.00	4.00	2.00	2.00	4.00	1

$$\lambda^*_{1,i}(\mathbf{p}) = (0.80, 5.00) \cdot 1 = (0.80, 5.00)$$
$$\lambda^*_{2,i}(\mathbf{p}) = (3.60, 4.50) \cdot 10/9 = (4.00, 5.00)$$
$$\lambda^*_{3,i}(\mathbf{p}) = (3.60, 4.50) \cdot 8/9 = (3.20, 4.00)$$
$$\lambda^*_{5,i}(\mathbf{p}) = (2.00, 2.00) \cdot 1 = (2.00, 2.00)$$

Decomposition as higher-order factor model - DGP



Estimation and Monte Carlo study

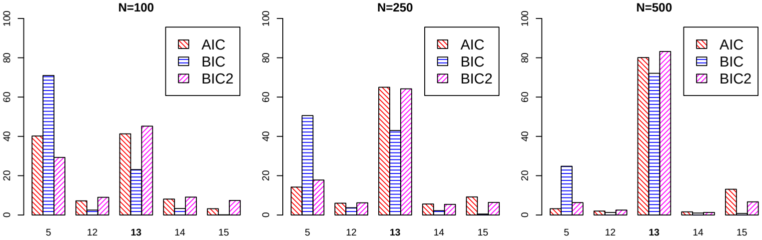
The hierarchical model can be used to search for the CIS structure

- Consider all partitions of the set of indicators
- For each partition set up an higher-order model
- Estimate the average loading profile and the static indicator loadings
- Scaling: Unit variances for the factors, zero variances for the CIS and effects scaling for the static loadings (cf., Proposition #2)
- Compare the models for each partition using information criteria (AIC, BIC, BIC2)

Monte Carlo study

- 15 partitions
- 10.000 replications of the example DGP
- Sample sizes: $N \in \{150, 250, 500\}$
- Estimation using lavaan

Results - CIS structure



- Partition 5: $\{\{2,3,4\},\{1\}\}$
- Partition 12: $\{\{2,4\},\{1\},\{3\}\}$
- **Partition 13:** $\{\{\mathbf{2,3}\}, \{\mathbf{1}\}, \{\mathbf{4}\}\}$
- Partition 14: $\{\{3,4\},\{1\},\{2\}\}$
- Partition 15: $\{\{1\},\{2\}\{3\},\{4\}\}$

- BIC2 work fairly well, even for a small sample size
- The effectiveness depends on the sample size and the measurement error (cf., Klopp & Klößner, 2022)

Results - Considerations

- We have a CIS structure, i.e., indicators that are commensurable and may be metrically invariant.
- Metric MI tests for each CIS.
- Testing the "invariance" of *only one* indicator is impossible (Klopp & Klößner, 2023).
- If indicators j_1 and j_2 are commensurable, there is a change of scale such that these indicators are metrically invariant (Klopp & Klößner, 2023; Klößner & Klopp, 2017, cf., Yoon & Millsap, 2007).

Loadings	Time 1	Time 2	Time 1	Time 2
λ_{1l}	0.80	5.00	0.20	1.00
λ_{2l}	4.00	5.00	1.00	1.00
λ_{3l}	3.20	4.00	0.80	0.80
λ_{4l}	2.00	2.00	0.50	0.40
Lat. var.	1.00	1.00	16.00	25.00
Lat. cov	0.24		4.80	

- Change of scale: $\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_1 \cdot \boldsymbol{D}$ and $\boldsymbol{\Phi}_2 = \boldsymbol{D}^{-1} \cdot \Phi_1 \cdot \boldsymbol{D}^{-1}$
- In the example: $\boldsymbol{D} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \end{pmatrix}$

Remarks

Advantages:

- Theoretically derived approach to find commensurable, i.e., potentially metrically invariant indicators.

Disadvantages:

- Brute force approach.
- Number of partitions grows fast (Bell numbers: 1, 2, 5, 15, 52, 203, 877, 4140, 21147, ...).
- Computationally demanding when there are a lot of indicators.

References I

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