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On the way to state specific response errors: A generalized local independence model

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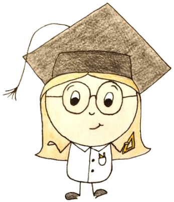
The Theory of Knowledge Structures

Basic concepts



- $Q = \{a, b, c, d\}$
- $K = \{a, b\}$
- $R = \{a, d\}$

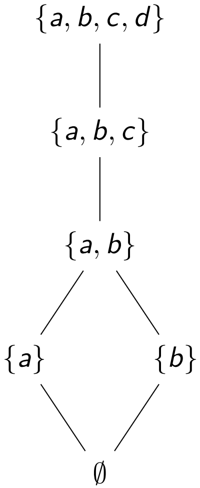


Figure: Knowledge structure $\mathcal{K}$ on domain Q.

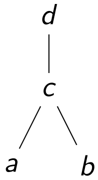


Figure: Corresponding precedence relation $\preccurlyeq\ \subseteq Q \times Q$.

Probabilistic Knowledge Structure Theory

The *basic local independence model (BLIM)* is given by:

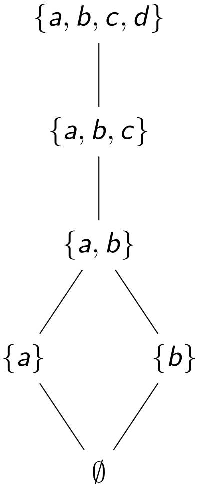
$$P(R) = \sum_{K \in \mathcal{K}} \pi_K \cdot P(R|K),$$

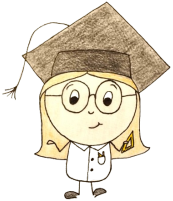
$$P(R|K) = \prod_{q \in K \setminus R} \beta_q \cdot \prod_{q \in K \cap R} (1 - \beta_q) \cdot \prod_{q \in R \setminus K} \eta_q \cdot \prod_{q \in Q \setminus (R \cup K)} (1 - \eta_q),$$

with the following parameters:

- β_q probability of an careless error at item $q \in Q$
- η_q probability of a lucky guess at item $q \in Q$
- π_K marginal probability of state $K \in \mathcal{K}$

Idea of a Generalized Local Independence Model





- $K = \{a\}$
- η_d small
- β_a big

- $K = \{a, b, c\}$
- η_d big
- β_a small

A Generalized Local Independence Model (GLIM)

$$
\begin{aligned}
P(R) &= \sum_{K \in \mathcal{K}} \pi_K \cdot P(R|K), \\
P(R|K) &= \prod_{q \in K \setminus R} w_\beta(d(q,K)) \cdot \beta_q \cdot \prod_{q \in K \cap R} (1 - w_\beta(d(q,K)) \cdot \beta_q) \\
&\quad \cdot \prod_{q \in R \setminus K} w_\eta(d(q,K)) \cdot \eta_q \cdot \prod_{q \in Q \setminus (R \cup K)} (1 - w_\eta(d(q,K)) \cdot \eta_q),
\end{aligned}
$$

- π_K, β_q, η_q BLIM parameter
- w_β, w_η non-negative, decreasing functions
- $d(q, K)$ a discrepancy between item q and knowledge state K

Item-State-Discrepancies

Layer discrepancy according to (Doble, Matayoshi, Cosyn, Uzun, & Karami, 2019)

Let $\mathcal{K}$ be a discriminative structure on Q and S a subset of Q. We define *first outer and inner layer*

$$S^{ol_1} := \{q \notin S \mid \forall p \notin S, p \preccurlyeq q \implies p = q\},$$
$$S^{il_1} := \{q \in S \mid \forall p \in S, q \preccurlyeq p \implies p = q\},$$

and the *n-th outer and inner layer* for $n \geq 2$

$$S^{ol_n} := (S \cup \bigcup_{i=1}^{n-1} S^{ol_i})^{ol_1},$$
$$S^{il_n} := (S \setminus \bigcup_{i=1}^{n-1} S^{il_i})^{il_1}.$$

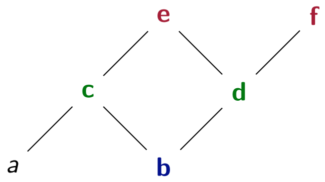
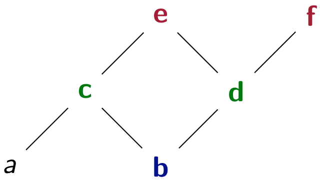
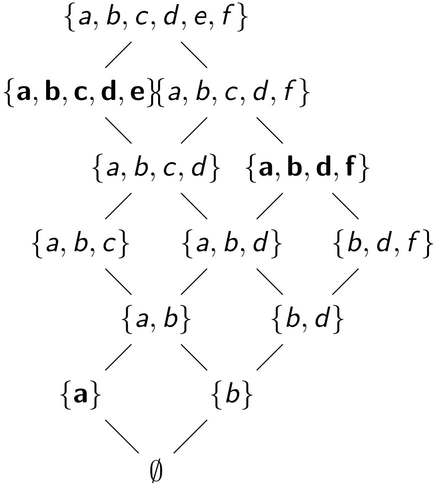


Figure: Example $d^{o}_{layer}(f, \{a\})$.

Item-State-Discrepancies

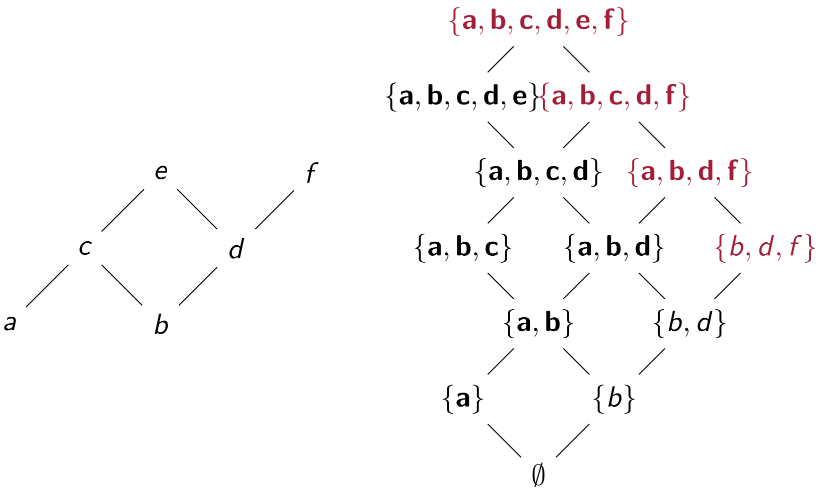
Example $d^{o}_{layer}(f, \{a\})$





Item-State-Discrepancies

Example $d^o_{min}(f, \{a\})$



Item-State-Discrepancies

Minimum Discrepancy

Let $\mathcal{K}$ be a knowledge structure on Q, $q \in Q$ and $K \in \mathcal{K}$. Then we define:

$$K_{q,\subseteq} := \{L \in \mathcal{K}_q \mid K \subseteq L\},$$
$$K_{\overline{q},\supseteq} := \{L \in \mathcal{K}_{\overline{q}} \mid K \supseteq L\}.$$

The *outer minimum discrepancy* is

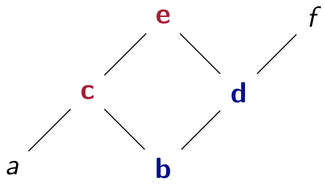
$$d^o_{min} : Q \times \mathcal{K}_{\overline{q}} \to \mathbb{N},\ (q, K) \mapsto d^o_{min}(q, K) := \min_{L \in K_{q,\subseteq}} \{|L \setminus K|\} \quad \text{for } q \notin K.$$

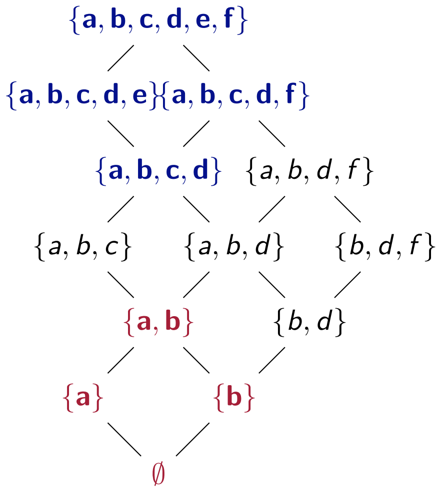
The *inner minimum discrepancy* is

$$d^i_{min} : Q \times \mathcal{K}_q \to \mathbb{N},\ (q, K) \mapsto d^i_{min}(q, K) := \min_{L \in K_{\overline{q},\supseteq}} \{|K \setminus L|\} \quad \text{for } q \in K.$$

Properties of the Item-State-Discrepancies

Example

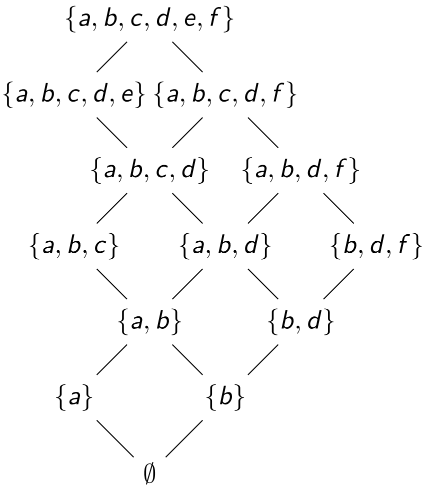




Simulation study

Data generation

- knowledge structure with 13 states and 6 items
- true parameters
  - $\beta_q \in [0.1, 0.25]$ for all $q \in Q$
  - $\eta_q \in [0.1, 0.25]$ for all $q \in Q$
  - $\lambda = 1$
  - $r_K \in [0, 1]; \pi_K = \frac{r_K}{\sum_{K \in \mathcal{K}} r_K}$ for all $K \in \mathcal{K}$
- function $w_\beta = w_\eta = exp(-\lambda \cdot d(q, K))$
- response patterns
  - $N = 7000$ response pattern were generated based on $\mathcal{K}$



Simulation study

Parameter estimation via Bayesian random sampling

Priors

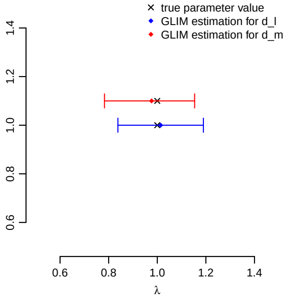
- $\beta_q \sim \mathit{beta}(2, 9)$ for all $q \in Q$
- $\eta_q \sim \mathit{beta}(2, 9)$ for all $q \in Q$
- $\pi_K \sim \mathit{dirichlet}(2, ..., 2)$ for all $K \in \mathcal{K}$
- $\lambda \sim \mathit{normal}(1, 3)$

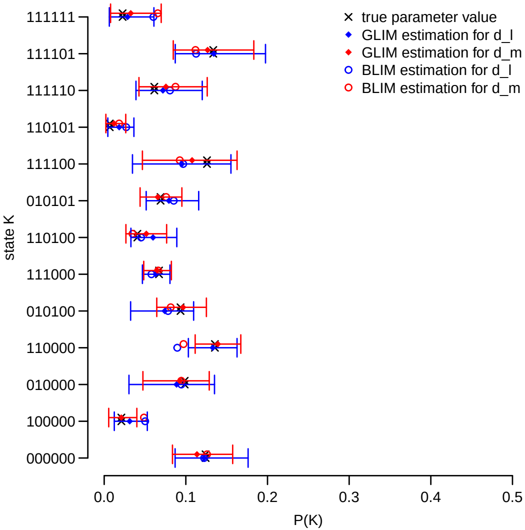
Random sampling

- 4 chains with 2000 samples
- 1000 samples per chain to initialize
- Algorithm: NUTS

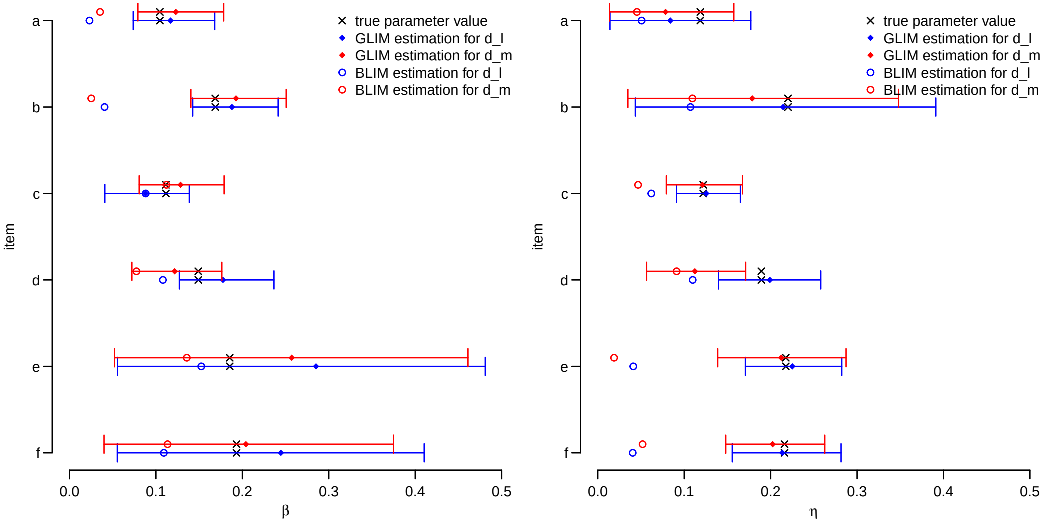
Simulation study

Parameter recovery





Simulation study



Thank you for your attention!

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Literature

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