EAM2025 XI Conference

23RD - 25TH JULY 2025

Spain Tenerife Canary Islands

European Association of Methodology

Estimating Context Effects in Small Samples while Controlling for Covariates: An Optimally Regularized Bayesian Estimator for Multilevel Latent Variable Models

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Agenda:

- Motivation
- Theory & Estimator
- Simulation Insights
- The **MLOB** R Package
- Example Application
- Discussion & Conclusion

Motivation

- Estimate between-group effects in multilevel data.
- Small samples or low ICCs pose challenges.
- Maximum likelihood can be unstable or biased.
- We propose an MSE-optimal regularized Bayesian estimator.
- Supports covariates for realistic modelling.
- Available on CRAN via the MLOB R package.

Theory: Univariate Model Estimation

Start from the multilevel latent variable models as denoted by Zitzmann et al. (2020, 2021):

$$\text{Level 1: } Y_{ij} = \beta_{0j} + \beta_w X_{w,ij} + \varepsilon_{ij} \quad (1)$$

$$\text{Level 2: } \beta_{0j} = \alpha + \beta_b X_{b,j} + \delta_j \quad (2)$$

Maximum likelihood (ML): vs. Regularized Bayesian:

$$\hat{\beta}_{b,\mathrm{ML}} = \frac{\hat{\tau}_{YX}}{\hat{\tau}_X^2} \quad (3) \qquad \hat{\beta}_{b,\mathrm{B}} = \frac{\hat{\tau}_{YX}}{(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2} \quad (4)$$

Theory: Extended Model Estimation

Introduce covariates into the multivariate latent variable model

(Dashuk et al., 2025):

$$\text{Level 1: } Y_{ij} = \beta_{0j} + \beta_w X_{w,ij} + C_{w,ij}\gamma + \varepsilon_{ij} \quad (5)$$

$$\text{Level 2: } \beta_{0j} = \alpha + \beta_b X_{b,j} + C_{b,j}\gamma + \delta_j \quad (6)$$

Extended regularized Bayesian estimator (more details):

$$\tilde{\beta}_b = \frac{\hat{\tau}_{YX} - \gamma'\tau_{CX}}{(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2} \quad (7)$$

Theory: Advantages of New Estimator

- Data-driven prior selection
  - Prior parameters ω and τ_0^2 automatically chosen to minimize MSE.
- Incorporates covariates
  - Allows for more realistic and flexible models.
- Bias–variance trade-off
  - Accepts small bias to achieve lower variance and overall MSE.
- Robust in challenging settings
  - Optimized performance in small samples and low ICCs.

Simulation. Data-Generating Process

540 cases covered with 5000 replications each:

- ICC_X: Intra-Class Correlation, {0.05, 0.1, 0.3, 0.5}.
- J: Number of groups, {5, 10, 20, 30, 40}.
- n: Number of individuals, {5, 15, 30}.
- k: Number of covariates, {1}, with γ: true covariate parameter, {0.3}.
- β_b: True between-group parameter {0.2, 0.5, 0.6}.
- β_w: True within-group parameter {0.2, 0.5, 0.7}.

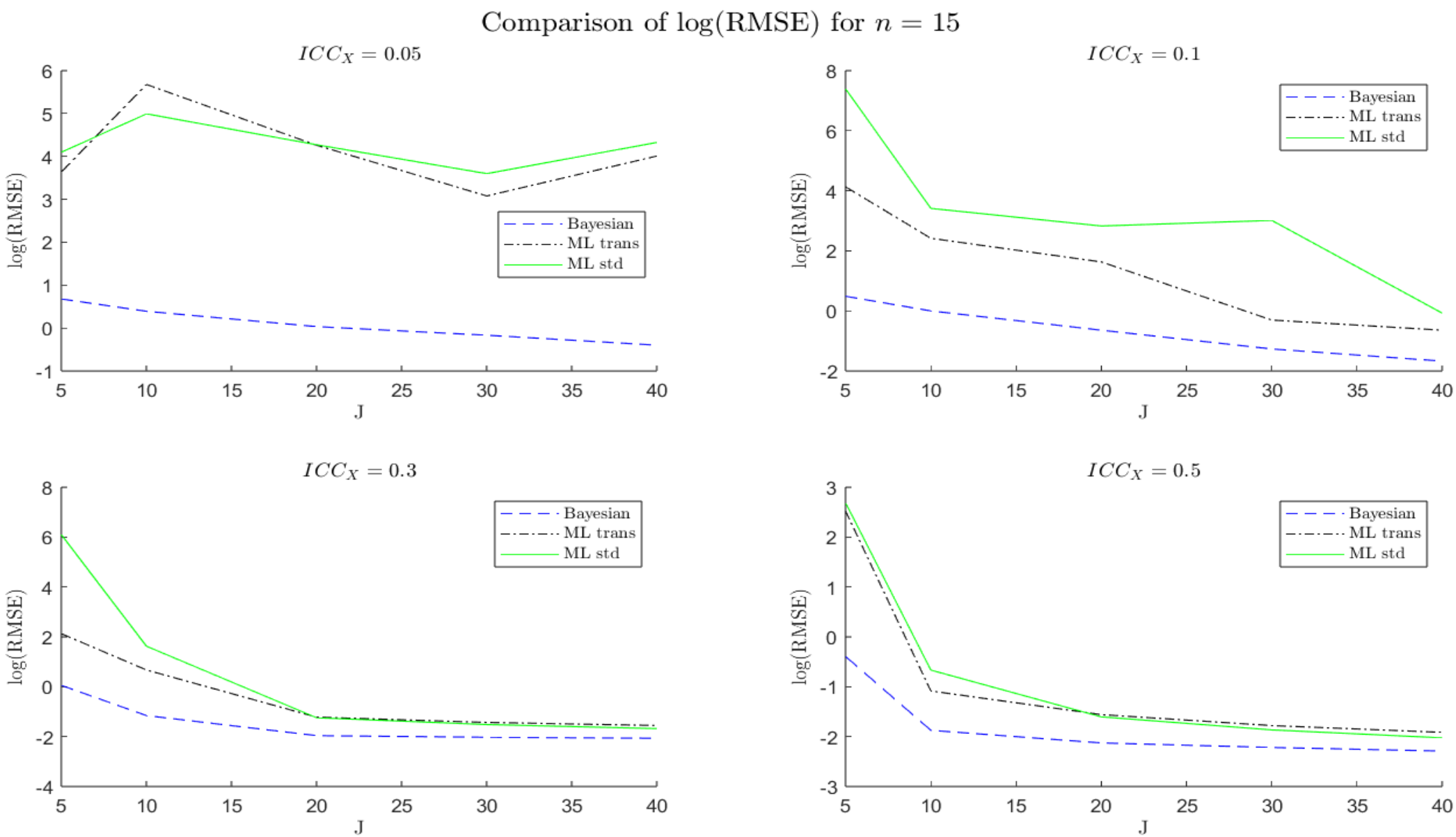
Simulation. Methods and Quality Evaluation

We estimated the generated data with 3 estimators:

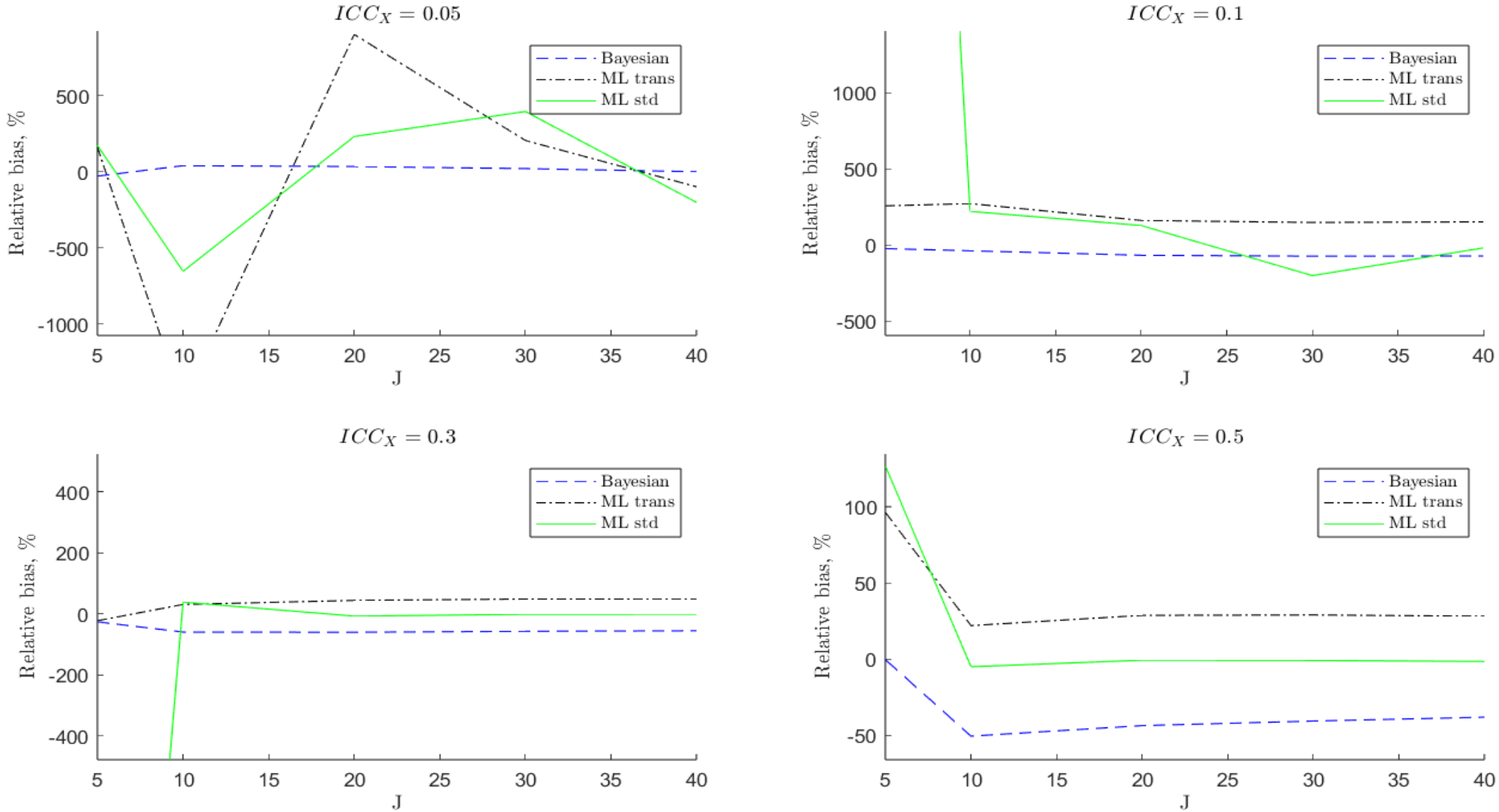
- Extended regularized Bayesian estimator.
- Standard ML estimator.
- Transformed ML estimator, applied to the transformed model.

Quality of estimators is measured with:

- Root mean squared error (RMSE).
- Coverage rate (CR).
- Relative bias (RB).
- Standard error ratio (SER).


Comparison of log(RMSE) for $n = 15$
$ICC_X = 0.05$
$ICC_X = 0.1$
$ICC_X = 0.3$
$ICC_X = 0.5$
log(RMSE)
J
Bayesian
ML trans
ML std

Comparison of relative bias in % for $n = 15$



MLOB Package: Key Features

To empower applied researchers use our estimator, we developed **MLOB** package in R:

- Estimates between-group effects using the extended regularized Bayesian approach.
- Handles unbalanced designs by effectively balancing it.
- Offers delete-d jackknife standard errors.
- Returns user-friendly output: estimates, SEs, CIs, Z-values, p-values.

MLOB Package: Usage in R

```
# Install

install.packages("MultiLevelOptimalBayes")

# Load

library(MultiLevelOptimalBayes)

# Fit model

result < - mlob(formula, data, group, balancing.limit, conf.level, jackknife,
punish.coeff)

summary(result)
```

Note: Dev. version of the MLOB package is available here:
https://github.com/MLOB-dev/MLOB

Real-World Application: Estimation

PASSNYC Dataset (1,272 schools, 32 Districts) – Estimating Math Proficiency

(https://www.kaggle.com/datasets/passnyc/data-science-for-good/data):

```
# Load and clean data

data <- read.table("2016 School Explorer.csv", sep=',', header=TRUE)

data_s <- data[data$Average.Math.Proficiency != 'N/A',]

data_s$math <- as.numeric(data_s$Average.Math.Proficiency)

data_s$ELA <- as.numeric(data_s$Average.ELA.Proficiency)

data_s$ENI <- as.numeric(data_s$Economic.Need.Index)

# Run extended regularized Bayesian model

result <- mlob(math ~ ENI + ELA + ENI:ELA, data = data_s, group = 'District’,
balancing.limit = 0.35)
```

Real-World Application: Results

```
➢ summary(result)

Call: mlob(math ~ ENI + ELA + ENI:ELA, data = data_subset, group = District, balancing.limit = 0.35)

Summary of Coefficients:

             Estimate Std. Error  Lower CI (95%) Upper CI (95%)  Z value   Pr(>|z|)    Significance
beta_b       -0.1169    0.01868    -0.1535        -0.0802        -6.2562   3.94e-10         ***
gamma_ELA     1.1324    0.21136     0.7182         1.5467         5.3580   8.41e-08         ***
gamma_ENI:ELA 0.3369    0.57599    -0.7920         1.4658         0.5849   5.58e-01

For comparison, summary of coefficients from unoptimized analysis (ML):

             Estimate Std. Error  Lower CI (95%) Upper CI (95%)  Z value   Pr(>|z|)    Significance
beta_b       -0.7721    0.3323     -1.4235        -0.1207        -2.3233   2.01e-02          *
gamma_ELA     1.1324    0.2113      0.7182         1.5467         5.3580   8.41e-08         ***
gamma_ENI:ELA 0.3369    0.5759     -0.7920         1.4658         0.5849   5.58e-01

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Limitations & Future Steps

Limitations:

- Advantage over ML diminishes with increasing sample size.
- Small risk of model misspecification under extreme conditions.
- Current model assumes two-level grouping only.

Future steps:

- Extend the model to more than two levels.
- Extend the model to handle directly unbalanced data.
- Extend the model to relax distributional assumptions.

Conclusion

- Introduced an **extended regularized Bayesian estimator** for between-group effects of multilevel latent variable model with covariates.
- Estimator is MSE-optimal: balances bias and variance.
- Handles small samples, low ICCs, and unbalanced data.
- Implemented in the **MLOB** R package with user-friendly interface.
- Validated with real-world data.
- **Try MLOB on your own data via CRAN!**

Sources & Funding

- Dashuk, V., Hecht, M., Lüdtke, O., Robitzsch, A. & Zitzmann, S. (2024). *An Optimally Regularized Estimator of Multilevel Latent Variable Models, with Improved MSE Performance*.
- Dashuk, V., Hecht, M., Lüdtke, O., Robitzsch, A. & Zitzmann, S. (2025). *Estimating Context Effects in Small Samples while Controlling for Covariates: An Optimally Regularized Bayesian Estimator for Multilevel Latent Variable Models*.
- Dashuk, V., Timilsina, B., Hecht, M. & Zitzmann, S. (2025). *MultiLevelOptimalBayes (MLOB): An R package for Regularized Bayesian Estimation of Multilevel Latent Variable Models with Covariates*.

- Funding: Funded by DFG (Project No. 471861494).

Thank you!

We invite and welcome your esteemed participation for any inquiries, reflections, and questions.

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Extended Regularized Bayesian Estimator 1

Model from Dashuk et al. (2025):

$$\text{Level 1: } Y_{ij} = \beta_{0j} + \beta_w X_{w,ij} + C_{w,ij}\gamma + \varepsilon_{ij} \quad (8)$$

$$\text{Level 2: } \beta_{0j} = \alpha + \beta_b X_{b,j} + C_{b,j}\gamma + \delta_j \quad (9)$$

Regress covariates C_ij on $X\_(b,j)$ and $X\_(w,ij)$:

$$\hat{C}_{ij,k} = \hat{\phi}_0 + \hat{\phi}_1 X_{b,j} + \hat{\phi}_2 X_{w,ij} \quad (10)$$

Compute residuals:

$$\tilde{C}_{ij,k} = C_{ij,k} - \hat{C}_{ij,k} \quad (11)$$

Extended Regularized Bayesian Estimator 2

Regress dependent variable Y_{ij} on all k residuals $\tilde{C}_{ij}$:

$$\hat{Y}_{ij} = \tilde{C}_{ij}\,\hat{\gamma} \tag{12}$$

Define $\tilde{Y}_{ij}$ as the difference:

$$\tilde{Y}_{ij} = Y_{ij} - \hat{Y}_{ij} \tag{13}$$

In the variable $\tilde{Y}_{ij}$ there is no variability associated with the covariates.

Substitute it for Y_{ij} in the two-level model:

$$\text{Level 1: } \tilde{Y}_{ij} = \beta_{0j} + \beta_w X_{w,ij} + \varepsilon_{ij} \tag{14}$$

$$\text{Level 2: } \beta_{0j} = \alpha + \beta_b X_{b,j} + \delta_j \tag{15}$$

Extended Regularized Bayesian Estimator 3

The model compress to the univariate case, therefore apply the regularized Bayesian estimator for the univariate model:

$$\tilde{\beta}_{b,\mathrm{B}} = \frac{\hat{\tau}_{\tilde{Y}X}}{(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2} = \frac{\hat{\tau}_{YX} - \gamma'\tau_{CX}}{(1-\omega)\tau_0^2 + \omega\hat{\tau}_X^2} \tag{16}$$

The estimator is derived, but how to choose priors ω and τ_0^2 to minimize MSE?

$$\mathrm{MSE}(\omega, \tau_0^2) = \mathrm{Var}\left(\tilde{\beta}_b(\omega, \tau_0^2)\right) + \left(\mathrm{Bias}\left(\tilde{\beta}_b(\omega, \tau_0^2)\right)\right)^2 \tag{17}$$

Extended Regularized Bayesian Estimator 4

Regularized Bayesian estimator expressed as F-distributed random variable:

$$\frac{\kappa_B(\omega,\tau_0^2)\theta_B(\omega,\tau_0^2)}{\kappa_2\theta_2}\,\tilde{\beta}_b \sim F(2\kappa_2, 2\kappa_B(\omega,\tau_0^2)) \qquad (18)$$

It depends on prior variance τ_0^2 and weight ω.

Priors are optimized by minimizing the MSE, expressed as a function of the prior parameters:

$$(\omega,\tau_0^2)_{optimal} = \arg\min_{(w,\tau_0^2)} \mathrm{MSE}(\tilde{\beta}_b\ (\omega,\tau_0^2)) \qquad (19)$$

■

←

Extended Regularized Bayesian Estimator 5

Sample covariances $\hat{\tau}_X^2$ and $\hat{\tau}_{\tilde{Y}X}$ can be calculated as:

$$\hat{\tau}_{\tilde{Y}X} = \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\bar{X}_j\,\bar{\tilde{Y}}_j - \frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n}X_{ij}\tilde{Y}_{ij} + \frac{J}{J-1}\,\bar{X}\,\bar{Y} \qquad (20)$$

$$\hat{\tau}_X^2 = \frac{nJ-1}{(n-1)(J-1)J}\sum_{j=1}^{J}\bar{X}_j{}^2 - \frac{1}{n(n-1)J}\sum_{j=1}^{J}\sum_{i=1}^{n}X_{ij}{}^2 + \frac{J}{J-1}\,\bar{X}^2 \qquad (21)$$

Extended Regularized Bayesian Estimator 6

The distributions of the sample covariances $\hat{\tau}_X^2$ and $\hat{\tau}_{\tilde{Y}X}$:

$$\hat{\tau}_X^2 \sim \text{Gamma}\left(\kappa_1 = \frac{\left(\sum_{i=1}^{nJ+J+1} \theta_{X,i}\right)^2}{2\sum_{i=1}^{nJ+J+1} \theta_{X,i}^2}, \theta_1 = \frac{\sum_{i=1}^{nJ+J+1} \theta_{X,i}^2}{\sum_{i=1}^{nJ+J+1} \theta_{X,i}}\right) \quad (22)$$

$$\hat{\tau}_{\tilde{Y}X} \sim \text{Gamma}\left(\kappa_2 = \frac{\left(\sum_{i=1}^{2(nJ+J+1)} \theta_{\tilde{Y}X,i}\right)^2}{2\sum_{i=1}^{2(nJ+J+1)} \theta_{\tilde{Y}X,i}^2}, \theta_2 = \frac{\sum_{i=1}^{2(nJ+J+1)} \theta_{\tilde{Y}X,i}^2}{\sum_{i=1}^{2(nJ+J+1)} \theta_{\tilde{Y}X,i}}\right) \quad (23)$$

With $\theta_{X,i}$ and $\theta_{\tilde{Y}X,i}$ - eigenvalues of transformed sums of $\hat{\tau}_X^2$ and $\hat{\tau}_{\tilde{Y}X}$

←

Extended Regularized Bayesian Estimator 7

The MSE of Bayesian estimation for between-group parameter β_b as a function of prior parameters τ_0^2 and ω:

$$MSE\ (\beta_b) = \frac{\kappa_2 * \theta_2^2 * (\kappa_B(\tau_0^2, \omega) + \kappa_2\ - 1)}{\theta_B^2(\tau_0^2, \omega)\ * (\kappa_B(\tau_0^2, \omega) - 1)^2 * (\kappa_B(\tau_0^2, \omega) - 2)} + \left(\frac{\kappa_2 * \theta_2}{(\kappa_B(\tau_0^2, \omega)\ - 1)\ * \theta_B(\tau_0^2, \omega)} - \beta_b\right)^2 \quad (24)$$

Equation (24) is further optimized for τ_0^2 and ω